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On Rearrangeable and Non-Blocking Switching Networks

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Switching networks of the type used in telephone exchanges are studied, with emphasis on a particular class of networks possessing great structural symmetry. This class contains rearrangeable networks carrying N calls with roughly $6N \log_3 N$ contacts, and non-blocking networks with roughly $16N(\log_5 N)^2$ contacts; these results are the best obtainable by the methods used. We also show, by an impractical argument, that there are non-blocking networks with roughly $90N \log_3 N$ contacts.

1. INTRODUCTION

Many parts of a telephone exchange have a cost proportional to the number of subscribers, but the switching network in a telephone exchange shows a diseconomy of scale: a network that serves twice as many subscribers costs more than twice as much. Thus as larger and larger exchanges are considered, the cost of the network becomes more and more important.

We shall study the problem of building a switching network for a telephone exchange with the minimum possible cost. This study will involve several idealizations that deserve discussion. First, we shall examine the problem from a combinatorial rather than a probabilistic point of view. Thus we shall seek networks without blocking rather than networks with a small probability of blocking. The questions studied combinatorially here are studied probabilistically in Pippenger [10, 11]. Though non-blocking networks are not used in practice to the extent that seldom-blocking networks are, they clearly would be preferable if they could be built at the same cost. Whether the apparent difference in cost between these two types of networks is real or illusory is an open question that will probably require further study of both types for its resolution.

Second, we assess the cost of a network simply as the number of contacts it contains, and concern ourselves not with its numerical value but only with its asymptotic behavior. These idealizations are traditional, and are used in most of the literature we cite. Furthermore, they (or other equally idealistic assumptions) are necessary if analysis is to be carried out; without them we would be reduced to tabulating the results of numerical optimizations. Thus they will enable us to observe a number of interesting qualitative phenomena against a background of confusing quantitative detail. Once these phenomena are understood, they can be used heuristically in the search for networks whose true costs are numerically optimal.

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2. Switching Networks

A switching network is a system for establishing simultaneous paths from terminals called *inputs* to other terminals called *outputs*. The paths are established through single-pole single-throw switches called *contacts*, which may interconnect not only the external terminals (inputs and outputs) but also internal terminals called *links*.

A network with an equal number of inputs and outputs will be called *square*, and in a square network the common number of inputs and outputs will be called the number of *lines*. Though our final results will always refer to square networks, non-square networks will arise at intermediate steps in their derivation.

We shall present some well-known constructions for networks. Our presentation will be rather informal; a more formal one can be found in Cantor [5].

The *crossbar* $\mathscr{C}(A, B)$ is a network with A inputs, B outputs, and a separate contact for connecting each input to each output (see Figure 1). It has AB contacts in all. The square crossbar $\mathscr{C}(M, M)$ will be denoted $\mathscr{C}(M)$. It has M^2 contacts.

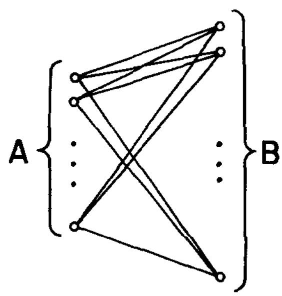

FIG. 1. The crossbar $\mathscr{C}(A, B)$. The lines represent contacts.

Suppose $\mathscr{F}$ is a network with I inputs and J outputs, and $\mathscr{F}'$ is a network with I' inputs and J' outputs. The *product* $\mathscr{F} \times \mathscr{F}'$ is a network obtained by taking I' copies of $\mathscr{F}$ and J copies of $\mathscr{F}'$, and by interconnecting them as shown in Figure 2. There is one link interconnecting each copy of $\mathscr{F}$ with each copy of $\mathscr{F}'$. The resulting network has II' inputs and JJ' outputs; if $\mathscr{F}$ has F contacts and $\mathscr{F}'$ has F', then $\mathscr{F} \times \mathscr{F}'$ has $FI' + JF'$.

The *open-face sandwich* $\mathscr{V}(A_1, B_1; \ldots; A_k, B_k)$ is the network $\mathscr{V}_k$ constructed according to the following scheme

$$\begin{aligned} \mathscr{V}_1 &= \mathscr{C}(A_1, B_1), \\ \mathscr{V}_2 &= \mathscr{V}_1 \times \mathscr{C}(A_2, B_2), \\ &\ldots \\ \mathscr{V}_k &= \mathscr{V}_{k-1} \times \mathscr{C}(A_k, B_k). \end{aligned}$$

It is easy to show that this network has

$$I = \prod_{1 \leqslant j \leqslant k} A_j$$

inputs,

$$J = \prod_{1 \leqslant j \leqslant k} B_j$$

outputs, and

$$V = \sum_{1 \leqslant j \leqslant k} \left(\prod_{1 \leqslant i \leqslant j} B_i \right) \left(\prod_{j \leqslant i \leqslant k} A_i \right)$$

contacts.

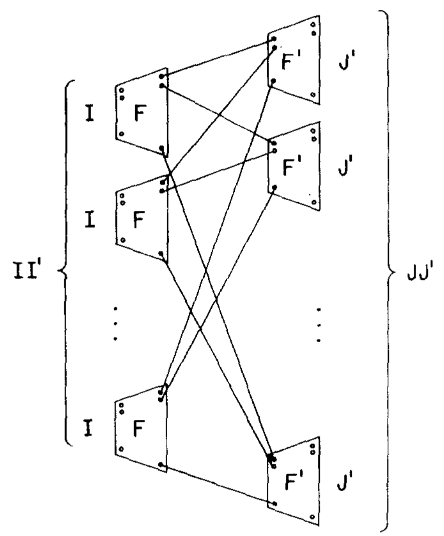

FIG. 2. The product $\mathscr{F} \times \mathscr{F}'$. The lines represent links.

Suppose $\mathscr{F}$ is a network with I inputs and J outputs and $\mathscr{G}$ is a square network with K lines. The *extended product* $\mathscr{F} \times\times \mathscr{G}$ is a network obtained by taking K copies of $\mathscr{F}$, J copies of $\mathscr{G}$, and K copies of the mirror-image $\mathscr{F}^*$ of $\mathscr{F}$, and by interconnecting them as shown in Figure 3. There is one link interconnecting each copy of $\mathscr{F}$ with each copy of $\mathscr{G}$, and one link interconnecting each copy of $\mathscr{G}$ with each copy of $\mathscr{F}^*$. The resulting network has IK lines; if $\mathscr{F}$ has F contacts and $\mathscr{G}$ has G, then $\mathscr{F} \times\times \mathscr{G}$ has $2FK + JG$.

The *sandwich* $\mathscr{W}(A_1, B_1; ...; A_k, B_k; M)$ is the network $\mathscr{W}_k$ constructed according to the following scheme.

$$\begin{aligned} \mathscr{W}_0 &= \mathscr{C}(M), \\ \mathscr{W}_1 &= \mathscr{C}(A_k, B_k) \times\times \mathscr{W}_0, \\ &\dots \\ \mathscr{W}_k &= \mathscr{C}(A_1, B_1) \times\times \mathscr{W}_{k-1}. \end{aligned}$$

It is easy to show that this network has

$$K = \left(\prod_{1 \leqslant j \leqslant k} A_j \right) M$$

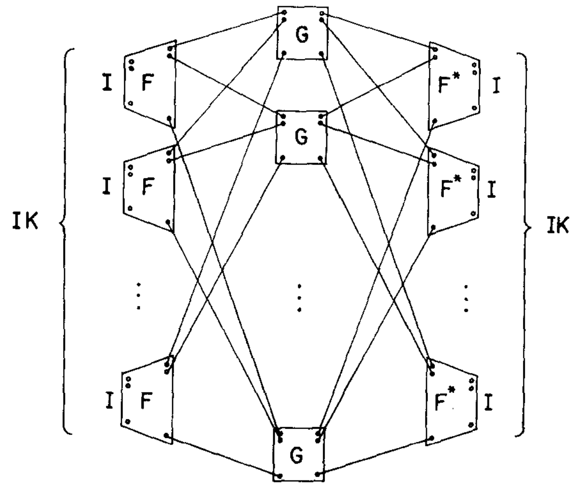

FIG. 3. The extended product $\mathscr{F} \times\times \mathscr{G}$. The lines represent links.

lines and

$$W = 2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i\leqslant j} B_i\right)\left(\prod_{j\leqslant i\leqslant k} A_i\right) M + \left(\prod_{1\leqslant j\leqslant k} B_j\right) M^2$$

contacts.

It is easy to see that, if the interconnections are suitably made, we have

$$\mathscr{F} \times (\mathscr{F}' \times \mathscr{F}'') \cong (\mathscr{F} \times \mathscr{F}') \times \mathscr{F}'',$$
$$\mathscr{F} \times\times (\mathscr{F}' \times\times \mathscr{G}) \cong (\mathscr{F} \times \mathscr{F}') \times\times \mathscr{G},$$

where $\cong$ means "is the same network as" (or, more formally, "is isomorphic to," as in Cantor [5]). From these facts we derive two propositions, both of which are easily established by induction on k.

PROPOSITION 2.1. $\mathscr{V}(A_1, B_1;...; A_k; B_k) \cong \mathscr{C}(A_1, B_1) \times \mathscr{V}(A_2, B_2;...; B_k, B_k)$.

PROPOSITION 2.2. $\mathscr{W}(A_1, B_1;...; A_k, B_k; M) \cong \mathscr{V}(A_1, B_1;...; A_k, B_k) \times\times \mathscr{C}(M)$.

The purpose of a network is to establish paths from inputs to outputs. We shall consider only paths of minimum length. In a crossbar these paths have length 1; in an open-face sandwich $\mathscr{V}(A_1, B_1;...; A_k, B_k)$ they have length k; in a sandwich $\mathscr{W}(A_1, B_1;...; A_k, B_k; M)$ they have length $2k + 1$. This minimum length will be called the number of *stages*.

At any moment in time, many paths may be established simultaneously, but no two can have a terminal in common (lest there be "crosstalk"). Such a set of paths will be called a *state*.

In any state, all the terminals that are involved in established paths will be called *busy*; all others will be called *idle*. A path will be called busy if any of the terminals involved in it is busy, and idle if all of them are idle.

We shall consider two tasks to be performed by networks. The networks performing these tasks are called ***rearrangeable*** and ***non-blocking***, respectively. A rearrangeable network satisfies the following condition: given a one-to-one correspondence between a set of inputs an equinumerous set of outputs, there exists a state in which each of these inputs is connected by a path to the corresponding output. A non-blocking network satisfies the following condition: given a state, an idle input, and an idle output, there exists an idle path connecting the input to the output. A non-blocking network is always rearrangeable, since any set of connections can be established one at a time; the converse is false, as the networks considered in the next section will show.

Throughout this paper we shall be asking questions of the form: what is the minimum possible number of contacts in rearrangeable or non-blocking networks satisfying certain conditions and having at least N lines? The constraint "at least N lines" (rather than "exactly N lines") is important when one of the conditions is that the network be a sandwich, since, for example, the only sandwiches with a prime number of lines are crossbars. We shall see in the following sections that sandwiches, despite their simplicity and structural symmetry, can perform our two tasks with as low a cost as any other networks for which explicit specifications have been given.

We shall use the following notations to indicate the asymptotic behavior of functions. The notation $O(f(N))$ will denote some function of N (possibly a different function at each occurrence) whose absolute value, when divided by $f(N)$, is ultimately bounded above by some positive constant. The notation $\Omega(f(N))$ is defined analogously, with "absolute" omitted and with "above" replaced by "below". Similarly, $o(f(N))$ will denote some functions of N whose absolute value, when divided by $f(N)$, tends to zero. The notation $\omega(f(N))$ is defined analogously, with "absolute" omitted and with "zero" replaced by "infinity". Roughly speaking, $O(f(N))$, $\Omega(f(N))$, $o(f(N))$, and $\omega(f(N)$ denote functions that grow at most as rapidly as, at least as rapidly as, less rapidly than, and more rapidly than $f(N)$. The notation $U(f(N))$ will denote a factor of the form $\exp O(f(N))$. Thus $U(1)$ denotes a function bounded between positive constants, and if $f(N) = o(1)$, then $U(f(N))$ is of the form $1 + O(f(N))$. Finally, we shall say that $f(N)$ is *asymptotic* to $g(N)$ if their ratio tends to unity. Since all our results concern asymptotic behavior, we need not worry if any of our arguments or constructions fail for the first few values of N.

3. Rearrangeable Networks

Rearrangeable networks can establish any set of connections from inputs to outputs. An additional request for connection, however, may require a complete rearrangement of the state. A request for disconnection, of course, presents no problems. Because of the great effort that may be required to satisfy a request, rearrangeable networks are not currently used in telephone exchanges, though they may find applications in other large-scale systems such as reconfigurable computers. It will be worthwhile for us to examine them, however, since this will display some useful techniques in a simple setting.

The crossbar $\mathscr{C}(N)$ is obviously a rearrangeable network with N lines and N^2 contacts. All extant methods for building rearrangeable networks with fewer contacts depend on

LEMMA 3.1. *If $\mathscr{F}$ is a rearrangeable network with I inputs and J outputs, if $\mathscr{G}$ is a rearrangeable network with K lines, and if $J \geqslant I$, then $F \times\times \mathscr{G}$ is a rearrangeable network with IK lines.*

The proof, which is a beautiful application of the matching (or marriage) theorem, can be found in Beneš [4] (Theorem 3.1).

If we consider the sandwich $\mathscr{W}(A_1, B_1;...; A_k; B_k; M)$ and apply Lemma 3.1 to the subnetworks $\mathscr{W}(A_k, B_k; M)$, $\mathscr{W}(A_{k-1}, B_{k-1}; A_k, B_k; M),..., \mathscr{W}(A_1, B_1;...;A_k, B_k; M)$ in turn we arrive at

CRITERION 3.2. *The sandwich $\mathscr{W}(A_1, B_1;...;A_k, B_k; M)$ is rearrangeable if $B_1 \geqslant A_1,..., B_k \geqslant A_k$.*

We shall prove two theorems on the cost of rearrangeable sandwiches.

THEOREM 3.3. *The minimum possible cost of a $(2k+1)$-stage sandwich with at least N lines that satisfies Criterion 3.2 is asymptotic to $\rho_k N^{1+1/(k+1)}$, where*

$$\rho_k = 2(k+1)(\tfrac{1}{2})^{1/(k+1)}.$$

Proof. We seek to minimize the cost,

$$W = 2 \sum_{1\leqslant j\leqslant k} \Big(\prod_{1\leqslant i\leqslant j} B_i\Big)\Big(\prod_{j\leqslant i\leqslant k} A_i\Big) M + \Big(\prod_{1\leqslant j\leqslant k} B_j\Big) M^2,$$

of the sandwich $\mathscr{W}(A_1, B_1;...; A_k, B_k; M)$ over integers such that

$$\Big(\prod_{1\leqslant j\leqslant k} A_j\Big) M \geqslant N,$$

$$B_j \geqslant A_j \qquad (1 \leqslant j \leqslant k).$$

It is clear that at the minimum

$$B_j = A_j \qquad (1 \leqslant j \leqslant k).$$

This, together with the first constraint, gives

$$W \geqslant \Big(2 \sum_{1\leqslant j\leqslant k} A_j + M\Big) N.$$

This can be minimized by droping the integrality constraints and using elementary calculus; the result is

$$W \geqslant 2(k+1)(N^{k+2}/2)^{1/(k+1)},$$

which is the desired lower bound.

To obtain an upper bound we set $A_j = Q$ for $1 \leqslant j \leqslant k$, $M = 2Q$, and of course $B_j = A_j$ for $1 \leqslant j \leqslant k$, where

$$Q = \lceil (N/2)^{1/(k+1)} \rceil.$$

These values satisfy the constraints and yield

$$W \leqslant 2(k+1)(N^{k+2}/2)^{1/(k+1)} U(N^{-1/(k+1)}),$$

which is the desired upper bound. ∎

THEOREM 3.4. *The minimum possible cost of a sandwich (of any number of stages) with at least N lines that satisfies Criterion 3.2 is asymptotic to $6N \log_3 N$.*

Proof. We seek to minimize

$$W = 2 \sum_{1 \leqslant j \leqslant k} \left(\prod_{1 \leqslant i \leqslant j} B_i \right) \left(\prod_{j \leqslant i \leqslant k} A_i \right) M + \left(\prod_{1 \leqslant j \leqslant k} B_j \right) M^2$$

over integers such that

$$\left(\prod_{1 \leqslant j \leqslant k} A_j \right) M \geqslant N,$$

$$B_j \geqslant A_j \qquad (1 \leqslant j \leqslant k),$$

where k is no longer constrained.

As in the proof of Theorem 3.3 we have

$$B_j = A_j \qquad (1 \leqslant j \leqslant k)$$

and

$$W \geqslant \left(2 \sum_{1 \leqslant j \leqslant k} A_j + M \right) N.$$

Since for any integer A,

$$\frac{A}{3 \log_3 A} \geqslant \frac{3}{3 \log_3 3} = 1,$$

we have

$$W \geqslant \left(6 \sum_{1 \leqslant j \leqslant k} \log_3 A_j + M \right) N.$$

This can be minimized by elementary calculus; the result is

$$W \geqslant 6N \log_3 N + O(N),$$

which is the desired lower bound.

To obtain an upper bound we set $A_j = 3$ for $1 \leqslant j \leqslant k$,

$$M = \lceil N/3^k \rceil,$$

and of course $B_j = A_j$ for $1 \leqslant j \leqslant k$, where

$$k = \lfloor \log_3 N - \tfrac{1}{2} \log_3 \log_2 N + \log_3 \tfrac{3}{2} \rfloor.$$

These values satisfy the constraints and yield

$$W \leqslant 6N \log_3 N + O(N(\log N)^{1/2}),$$

which is the desired upper bound. ∎

That rearrangeable networks can be built with cost $O(N \log N)$ was apparently observed first by Beizer [3], who used a network based on $\mathscr{C}(2)$ rather than $\mathscr{C}(3)$. This network was rediscovered by Beneš [4] (Chapter 3), Joel [8] and Waksman [12]. The advantage of using $\mathscr{C}(3)$ rather than $\mathscr{C}(2)$ is slight: a coefficient of 6 rather than $4 \log_2 3 = 6.339\ldots$.

4. Non-Blocking Networks

Non-blocking networks, like rearrangeable networks, can establish any set of connections from inputs of outputs. In contrast, however, an additional request for connection can be satisfied without disturbing existing connections and irrespective of which state the history of connections and disconnections has left the network in. Non-blocking networks are thus ideal telephone exchanges. We shall examine them in the remainder of this paper.

The crossbar $\mathscr{C}(N)$ is obviously a non-blocking network with N lines and N^2 contacts. All extant methods of building non-blocking networks with fewer contacts depend on the notion of a majority-access network. Consider a network in some state. We shall say that an input has access to an output if there is an idle path connecting them. A network is a majority-access network if, in any state, each idle input has access to more than half of the outputs. The point of this notion consists in

Lemma 4.1. *If $\mathscr{F}$ is a majority-access network and $\mathscr{G}$ is a non-blocking network, then $\mathscr{F} \times\times \mathscr{G}$ is a non-blocking network.*

The proof, which is a simple application of the pigeon-hole principle, is implicit in Clos [7].

It is easy to see that the crossbar $\mathscr{C}(A, B)$ is a majority-access network if and only if $B > 2(A - 1)$: an idle input has access to any idle output; there may be as many as $A - 1$ busy outputs, and these must constitute less than half the outputs.

If we consider the sandwich $\mathscr{W}(A_1, B_1;\ldots; A_k, B_k; M)$ and apply Lemma 4.1 to the subnetworks $\mathscr{W}(A_k, B_k; M), \mathscr{W}(A_{k-1}, B_{k-1}; A_k, B_k; M),\ldots, \mathscr{W}(A_1, B_1;\ldots; A_k, B_k; M)$ in turn, we arrive at

CRITERION 4.2. *The sandwich* $\mathscr{W}(A_1, B_1; \ldots; A_k, B_k; M)$ *is non-blocking if* $B_1 > 2(A_1 - 1), \ldots, B_k > 2(A_k - 1)$.

THEOREM 4.3. *The minimum possible cost of a* $(2k+1)$*-stage sandwich with at least* N *lines that satisfies Criterion* 4.2 *is asymptotic to* $\sigma_k N^{1+1/(k+1)}$, *where*

$$\sigma_k = 2(k+1)\left(\frac{2^{\binom{k+2}{2}}}{4}\right)^{1/(k+1)}.$$

Proof. We seek to minimize the cost

$$W = 2 \sum_{1 \leqslant j \leqslant k} \left(\prod_{1 \leqslant i < j} B_i\right)\left(\prod_{j \leqslant i \leqslant k} A_i\right) M + \left(\prod_{1 \leqslant j \leqslant k} B_j\right) M^2$$

of the sandwich $\mathscr{W}(A_1, B_1; \ldots; A_k, B_k; M)$ over integers such that

$$\left(\prod_{1 \leqslant j \leqslant k} A_j\right) M \geqslant N,$$

$$B_j > 2(A_j - 1) \qquad (1 \leqslant j \leqslant k).$$

It is clear that at the minimum

$$B_j = 2A_j - 1 \qquad (1 \leqslant j \leqslant k).$$

This, together with the first constraint, gives

$$W \geqslant \left(2 \sum_{1 \leqslant j \leqslant k} \left(\prod_{1 \leqslant i < j} \left(\frac{2A_i - 1}{A_i}\right)\right) A_j + \left(\prod_{1 \leqslant j \leqslant k} \left(\frac{2A_j - 1}{A_j}\right)\right) M\right) N.$$

For the purpose of proving the lower bound we may assume that $W = O(N^{1+1(k+1)})$. Let us show that we may also assume that $A_1, \ldots, A_k$ and M are all $N^{1/(k+1)} U(1)$. The obvious $2A_j - 1 \geqslant A_j$ for $j = 1, \ldots, k$, together with the first constraint, imply

$$W \geqslant \left(2 \sum_{1 \leqslant j \leqslant k} A_j + M\right) N.$$

From this it is clear that $A_1, \ldots, A_k$ and M must all be $O(N^{1/(k+1)})$, else we should not have $W = O(N^{1+1/(k+1)})$. These conditions, together with the first constraint, imply that $A_1, \ldots, A_k$ and M must all be $\Omega(N^{1/(k+1)})$. Thus they must all be $N^{1/(k+1)} U(1)$.

In view of the rate of growth of $A_1, \ldots, A_k$, the lower bound for W can be rewritten as

$$W \geqslant \left(2 \sum_{1 \leqslant j \leqslant k} 2^j A_j + 2^k M\right) N U(N^{-1/(k+1)}).$$

This can be minimized by elementary calculus; the result is

$$W \geqslant 2(k+1)\left(\frac{2^{\binom{k+2}{2}}N^{k+2}}{4}\right)^{1/(k+1)} U(N^{-1/(k+1)}),$$

which is the desired lower bound.

To obtain an upper bound we set $A_j = 2^{k-j}Q$ for $1 \leqslant j \leqslant k$, $M = 2Q$, and of course $B_j = 2A_j - 1$ for $1 \leqslant j \leqslant k$, where

$$Q = \left\lceil \frac{1}{2^k}\left(\frac{2^{\binom{k+2}{2}}N}{4}\right)^{1/(k+1)} \right\rceil.$$

These values satisfy the constraints and yield

$$W \leqslant 2(k+1)\left(\frac{2^{\binom{k+2}{2}}N^{k+2}}{4}\right)^{1/(k-1)} U(N^{-1/(k+1)}),$$

which is the desired upper bound. ▮

This method of exploiting Criterion 3.2 was used by Clos [7], who obtained not

$$\sigma_k = 2(k+1)\left(\frac{2^{\binom{k+2}{2}}}{4}\right)^{1/(k+1)},$$

but the somewhat larger

$$\sigma_k' = 5 \cdot 2^k - 4.$$

For purposes of comparison, we give the first few values of these sequences.

k	σ_k	σ_k'
1	5.656	6
2	15.119	16
3	32.000	36
4	60.628	76
5	107.756	156
6	166.431	316

A similar result can be obtained when the number of stages is not constrained. The cost is $N^{1+o(1)}$, but $N(\log N)^{\omega(1)}$. We shall not do this, since there are better ways of exploiting Criterion 4.2 when the number of stages is large. The best way currently available is a recursive method due to Cantor [5], which yields non-blocking networks with cost $O(N(\log N)^{\beta})$, where $\beta = 2.269...$ is the unique positive root of the transcendental equation $1 = 2^{1/(1-\beta)} + 3^{1/(1-\beta)}$. It is possible to do even better, however, without going beyond sandwiches.

We shall say that a network is an H-access network if, in any state, each idle input has access to at least H outputs. The point of this notion consists in

LEMMA 4.4. *If $\mathscr{F}$ is an H-access network with I inputs and J outputs and $\mathscr{F}'$ is a non-blocking network with I' inputs and J' outputs, then $\mathscr{F} \times \mathscr{F}'$ is an H'-access network, where $H' = HJ' - I(I' - 1)$.*

The proof can be found in Cantor [5] (Lemma 4.1). (An H-access network is, in Cantor's terminology, a network of type $T(1, H)$.)

CRITERION 4.5. *The open-face sandwich $\mathscr{V}(A_1, B_1;...; A_k, B_k)$ is a majority-access network (and thus the sandwich $\mathscr{W}(A_1, B_1;...; A_k, B_k; M)$ is non-blocking) if*

$$\prod_{1\leqslant j\leqslant k} B_j > 2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i<j} A_j\right)(A_i - 1)\left(\prod_{j<i\leqslant k} B_i\right).$$

Proof. For $1 \leqslant h \leqslant k$, let

$$\mathscr{V}_h = \mathscr{V}(A_1, B_1;...; A_h, B_h).$$

It is easy to show, using Lemma 4.4 inductively, that $\mathscr{V}_h$ is an H_h-access network with

$$I_h = \prod_{1\leqslant j\leqslant h} A_j$$

inputs and

$$J_h = \prod_{1\leqslant j\leqslant h} B_j$$

outputs, where

$$H_h = \prod_{1\leqslant j\leqslant h} B_j - \sum_{1\leqslant j\leqslant h} \left(\prod_{1\leqslant i<j} A_i\right)(A_j - 1)\left(\prod_{j<i\leqslant h} B_i\right).$$

The criterion thus asserts that H_k is more than half J_k, that is, that $\mathscr{V}_k$ is a majority-access network. ▮

THEOREM 4.6. *The minimum possible cost of a $(2k + 1)$-stage sandwich with at least N lines that satisfies Criterion 4.5 is asymptotic to $\tau_k N^{1+1/(k+1)}$, where*

$$\tau_k = 4(k + 1)^2(\tfrac{1}{8})^{1/(k+1)}.$$

Proof. We seek to minimize the cost,

$$W = 2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i\leqslant j} B_i\right)\left(\prod_{j\leqslant i\leqslant k} A_i\right) M + \left(\prod_{1\leqslant j\leqslant k} B_j\right) M^2,$$

of the sandwich $\mathscr{W}(A_1, B_1; \ldots; A_k, B_k; M)$ over integers such that

$$\left(\prod_{1\leqslant j\leqslant k} A_j\right) M \geqslant N,$$

$$\prod_{1\leqslant j\leqslant k} B_j > 2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i<j} A_i\right)(A_j - 1)\left(\prod_{j<i\leqslant k} B_i\right).$$

The first constraint gives

$$W \geqslant \left(2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i\leqslant j} \frac{B_i}{A_i}\right) A_j + \left(\prod_{1<j\leqslant k} \frac{B_j}{A_j}\right) M\right) N.$$

The second constraint leads easily to

$$\prod_{1\leqslant i\leqslant j} B_i \geqslant \prod_{1\leqslant i\leqslant j} A_i \qquad (1 \leqslant j \leqslant k).$$

This, together with the first constraint, gives

$$W \geqslant \left(2 \sum_{1\leqslant j\leqslant k} A_j + M\right) N.$$

As in the proof of Theorem 4.3, this implies that $A_1, \ldots, A_k$ and M must all be $N^{1/(k+1)} U(1)$.

The second constraint can be rewritten as

$$\frac{B_1}{A_1} > 2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1<i<j} \frac{A_i}{B_i}\right)\left(\frac{A_j - 1}{B_j}\right).$$

In view of the rate of growth of $A_1, \ldots, A_k$ this implies

$$\frac{B_1}{A_1} \geqslant 2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1<i\leqslant j} \frac{A_i}{B_i}\right) U(N^{-1/(k+1)}).$$

Substituting this into the lower bound for W yields

$$W \geqslant 2\left(\sum_{1\leqslant j\leqslant k} \left(\prod_{1<i\leqslant j} \frac{A_i}{B_i}\right)\right)\left(2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1<i\leqslant j} \frac{B_i}{A_i}\right) A_j + \left(\prod_{1<j\leqslant k} \frac{B_j}{A_j}\right) M\right) N U(N^{-1/(k+1)}).$$

This can be minimized by elementary calculus; the calculations are simplified by eliminating $B_2, \ldots, B_k$ in favor of the new variables $\vartheta_2, \ldots, \vartheta_k$ defined by

$$\vartheta_j = \prod_{1<i\leqslant j} \frac{B_i}{A_i}.$$

The result is

$$W \geqslant 4(k+1)^2 \left(\frac{N^{k+2}}{8}\right)^{1/(k+1)} U(N^{-1/(k+1)}),$$

which is the desired lower bound.

To obtain an upper bound we set $A_j = Q$ for $1 \leqslant j < k$, $A_k = 2Q$, $M = 4Q$, $B_1 = 2(k+1)Q$, and $B_j = Q$ for $1 < j \leqslant k$, where

$$Q = \lceil (N/8)^{1/(k+1)} \rceil.$$

These values satisfy the constraints and yield

$$W \leqslant 4(k+1)^2 (N^{k+2}/8)^{1/(k+1)} U(N^{-1/(k+1)}),$$

which is the desired upper bound. ∎

Most plans for non-blocking networks show a "midriff bulge," with the number of links interconnecting successive stages increasing toward the midpoint. But the networks found in the preceding theorem show a sifferent silhouette: not only is the number of links interconnecting successive stages constant through most of the network, but there is actually a constriction in the number of links interconnecting the central stage. This curious phenomenon shows the danger making *a priori* assumptions about the structure of optimal networks.

It is interesting to compare τ_k with σ_k. We give the first few values of these sequences.

k	τ_k	σ_k
1	5.656	5.656
2	18.000	15.119
3	38.054	32.000
4	65.975	60.628
5	101.823	107.756
6	145.627	166.431

We see that τ_k is no better than σ_k until $k = 5$, which corresponds to 11 stages, a value just beyond those used in current practice. Once the crossover occurs, however, τ_k quickly asserts its advantage over σ_k, since the former grows quadratically, and the latter exponentially. This ultimate advantage reveals itself in

THEOREM 4.7. *The minimum possible cost of a sandwich (of any number of stages) with at least N lines that satisfies Criterion* 4.5 *is asymptotic to* $16N(\log_5 N)^2$.

Proof. We seek to minimize

$$W = 2 \sum_{1 \leqslant j \leqslant k} \left(\prod_{1 \leqslant i \leqslant j} B_i\right)\left(\prod_{j \leqslant i \leqslant k} A_i\right) M + \left(\prod_{1 \leqslant j \leqslant k} B_j\right) M^2$$

over integers such that

$$\left(\prod_{1\leqslant j\leqslant k} A_j\right) M \geqslant N,$$

$$\prod_{1\leqslant j\leqslant k} B_j > 2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i<j} A_i\right)(A_j - 1)\left(\sum_{j<i\leqslant k} B_i\right),$$

where k is no longer constrained.

The second constraint easily leads to

$$\prod_{1\leqslant j\leqslant k} B_j \geqslant \prod_{1\leqslant j\leqslant k} A_j .$$

This, together with the first constraint, implies

$$W \geqslant \left(2B_1 \sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i<j} \frac{B_{i+1}}{A_i}\right) + M\right) N.$$

The second constraint can be rewritten as

$$B_1 > 2 \sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i<j} \frac{A_i}{B_{i+1}}\right)(A_j - 1).$$

Substituting this into the lower bound for W yields

$$W \geqslant \left(4\left(\sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i<j} \frac{A_i}{B_{i+1}}\right)(A_j - 1)\right)\left(\sum_{1\leqslant j\leqslant k} \left(\prod_{1\leqslant i<j} \frac{B_{i+1}}{A_i}\right)\right) + M\right) N.$$

Applying Cauchy's inequality,

$$\left(\sum_{1\leqslant j\leqslant k} x_j^2\right)\left(\sum_{1\leqslant j\leqslant k} y_j^2\right) \geqslant \left(\sum_{1\leqslant j\leqslant k} x_j y_j\right)^2,$$

we obtain

$$W \geqslant \left(4\left(\sum_{1\leqslant j\leqslant k} (A_j - 1)^{1/2}\right)^2 + M\right) N.$$

Since for any integer A,

$$\frac{(A-1)^{1/2}}{2\log_5 A} \geqslant \frac{(5-1)^{1/2}}{2\log_5 5} = 1,$$

we have

$$W \geqslant \left(4\left(2 \sum_{1\leqslant j\leqslant k} \log_5 A_j\right)^2 + M\right) N.$$

This can be minimized by elementary calculus; the result is

$$W \geqslant 16N(\log_5 N)^2 + O(N \log N \log \log N),$$

which is the desired lower bound.

To obtain an upper bound we set $A_j = 5$ for $1 \leqslant j \leqslant k$,

$$M = \lceil N/5^k \rceil,$$

$B_1 = 8k + 1$, and $B_j = 5$ for $1 < j \leqslant k$, where

$$k = \lfloor \log_5 N - \tfrac{1}{2} \log_5 \log_2 N + \log_5 \tfrac{5}{2} \rfloor.$$

These values satisfy the constraints and yield

$$W \leqslant 16N(\log_5 N)^2 + O(N(\log N)^{3/2}),$$

which is the desired upper bound. ∎

That non-blocking networks can be built with cost $O(N(\log N)^2)$ was first shown by Cantor [5, 6], who used a network based on $\mathscr{C}(2)$ rather than $\mathscr{C}(5)$. The advantage of using $\mathscr{C}(5)$ rather than $\mathscr{C}(2)$ is coefficient of 16 rather than $4(\log_2 5)^2 = 21.565\ldots$. (The asymptotic formula $2N(\log_2 N)^2$ given in [6] is erroneous; the correct formula is $4N(\log_2 N)^2$.)

5. Non-Blocking Networks Again

In preceding sections we have found rearrangeable networks with asymptotically $6N \log_3 N$ contacts and non-blocking networks with asymptotically $16N(\log_5 N)^2$ contacts. The former result is as good as any currently known, and the latter is as good as any currently known by constructive methods (that is, with an explicit specification being given for the network). But is known, by non-constructive methods, that there are non-blocking networks with $O(N \log N)$ contacts. In this section we shall prove

Theorem 5.1. *There are non-blocking networks with at least N lines and cost asymptotic to $90N \log_3 N$.*

The first step of the proof is a lemma concerning "sparse crossbars". (This felicitous terminology is due to G. M. Masson.) A sparse crossbar is like a crossbar, but has only a small fraction of the contacts. The lemma we need is

Lemma 5.2. *For every integer $k \geqslant 0$ there exists a sparse crossbar $\mathscr{S}_k$ with $3 \cdot 3^k$ inputs, $9 \cdot 3^k$ outputs, and cost at most $45 \cdot 3^k$, such that any set of 3^k inputs is joined by contacts to at least $6 \cdot 3^k$ different outputs.*

Proof. Let $K = 3^k$. (The proof will not use the fact that K is a power of 3.) Let p be a permutation on the set $\mathscr{K} = \{0, 1, \ldots, 45K - 1\}$. From p we obtain a sparse crossbar $\mathscr{S}(p)$

having $\{0, 1,..., 3K-1\}$ as inputs and $\{0, 1,..., 9K-1\}$ as outputs, and having a contact joining the input ($x \bmod 3K$) to the output ($p(x) \bmod 9K$) for every x in $\mathscr{K}$.

We shall say that a sparse crossbar $\mathscr{S}(p)$ is "good" if there do not exist a set $\mathscr{A}$ of K inputs and a set $\mathscr{B}$ of $6K$ outputs such that every contact that joins an input in $\mathscr{A}$ also joins an output in $\mathscr{B}$; we shall say that it is "bad" otherwise. A good sparse crossbar clearly satisfies the requirements of the lemma. We shall show that one exists by obtaining an upper bound less than unity on the fraction of all permutations p for which $\mathscr{S}(p)$ is bad.

Any set of $\mathscr{A}$ of K inputs corresponds to a set $\mathscr{I}$ of $15K$ elements of $\mathscr{K}$, and any set $\mathscr{B}$ of $6K$ outputs corresponds to a set $\mathscr{J}$ of $30K$ elements of $\mathscr{K}$. Every contact of $\mathscr{S}(p)$ that joins an input in $\mathscr{A}$ will also join an output in $\mathscr{B}$ if and only if p sends every element of $\mathscr{I}$ into $\mathscr{J}$. Of the $(45K)!$ permutations of $\mathscr{K}$, there are $[30K]_{15K}(30K)!$ that satisfy this condition, where $[N]_R = N(N-1)\cdots(N-R+1)$. There are

$$\binom{3K}{K}$$

possible choices for $\mathscr{A}$ and

$$\binom{9K}{6K} = \binom{9K}{3K}$$

possible choices for $\mathscr{B}$. Thus an upper bound on the fraction of all permutations p for which $\mathscr{S}(p)$ is bad is

$$\binom{3K}{K}\binom{9K}{3K}\frac{[30K]_{15K}(30K)!}{(45K)!} = \frac{\binom{3K}{K}\binom{9K}{3K}\binom{30K}{15K}}{\binom{45K}{15K}}.$$

We observe that

$$\binom{45K}{15K} \geqslant \binom{3K}{K}\binom{9K}{3K}\binom{33K}{11K},$$

since the number of ways of choosing $15K$ out of $45K$ objects is not less than the number of ways of choosing K out of the first $3K$, $3K$ out the next $9K$, and $11K$ out of the last $33K$. Thus the fraction of bad permutations is at most

$$\binom{30K}{15K}\Big/\binom{33K}{11K}.$$

This is easily shown to be less than unity for all K by use of Stirling's formula. ∎

The next step of the proof is to combine sparse crossbars to obtain majority-access networks. For technical reasons, a property stronger than majority-access will be used. We shall prove

LEMMA 5.3. *For every integer $k \geqslant 0$ there exists a $5\cdot 3^k$-access network $\mathscr{T}_k$ with 3^k inputs, $9\cdot 3^k$ outputs, and cost at most $(45k+5)\,3^k$.*

Proof. There is certainly a 5-access network $\mathcal{T}_0$ with 1 input, 9 outputs, and cost at most 5. We shall assume that $\mathcal{T}_k$ exists and build $\mathcal{T}_{k+1}$. We do this by taking 3 copies of $\mathcal{T}_k$ and identifying their outputs with the inputs of the sparse crossbar $\mathcal{S}_{k+1}$, as shown in Fig. 4.

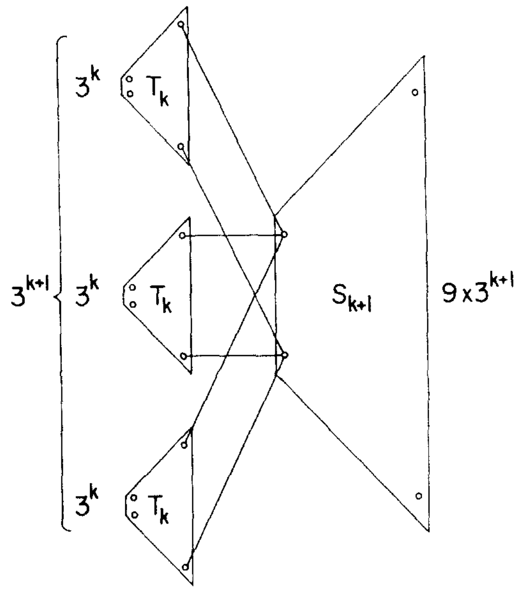

FIG. 4. The network $\mathcal{T}_{k+1}$. The lines represent links.

The 3 copies of $\mathcal{T}_k$ each have cost at most $(45k + 5)\, 3^k$, and $\mathcal{S}_{k+1}$ has cost at most $45 \cdot 3^{k+1}$. Thus $\mathcal{T}_{k+1}$ has cost at most $3(45k + 5)\, 3^k + 45 \cdot 3^{k+1} = (45(k+1) + 5)\, 3^{k+1}$.

It remains to show that an idle input of $\mathcal{T}_{k+1}$ has access to at least $5 \cdot 3^{k+1}$ outputs. Consider first only those calls in the same copy of $\mathcal{T}_k$. Despite these calls, the idle input has access to at least $5 \cdot 3^k$ inputs of $\mathcal{S}_{k+1}$, by definition of $\mathcal{T}_k$. Consider now the calls in the other 2 copies of $\mathcal{T}_k$. There are at most $2 \cdot 3^k$ such calls, since each copy has only 3^k inputs. Thus, despite these calls, the idle input has access to at least $3 \cdot 3^k = 3^{k+1}$ inputs of $\mathcal{S}_{k+1}$. By definition of $\mathcal{S}_{k+1}$, these 3^{k+1} inputs of $\mathcal{S}_{k+1}$ are joined by contacts to at least $6 \cdot 3^{k+1}$ different outputs of $\mathcal{S}_{k+1}$. At most 3^{k+1} of these outputs can be busy, since $\mathcal{T}_{k+1}$ has only 3^{k+1} inputs. Thus an idle input has access to at least $5 \cdot 3^{k+1}$ outputs. ∎

We are now ready for

Proof of Theorem 5.1. Consider the network

$$\mathcal{U}_k(M) = \mathcal{T}_k \times\times \mathcal{C}(M).$$

Since $\mathcal{T}_k$ is a majority-access network, $\mathcal{U}_k(M)$ is strictly non-blocking, by Lemma 4.1. It has $3^k M$ inputs, $3^k M$ outputs, and cost

$$U = 2(45k + 5)\, 3^k M + 9 \cdot 3^k M^2.$$

We set

$$k = \lfloor \log_3 N - \tfrac{1}{2} \log_3 \log_2 N + \log_3 \tfrac{3}{2} \rfloor,$$
$$M = \lceil N/3^k \rceil.$$

Then the number of lines is

$$3^k M \geqslant N$$

and

$$U \leqslant 90N \log_3 N + O(N(\log N)^{1/2}).$$

This completes the proof. ∎

That non-blocking networks can be built with cost $O(N \log N)$ was first proved by Bassalygo and Pinsker [2]. Their method, with $\alpha = 1/3$, $\beta = 2/3$, and $k = 4$, gives the asymptotic formula $66N \log_2 N$. The preceding theorem, which is proved by a modification of their method, yields a coefficient of 90 rather than $66 \log_2 3 = 104.607...$. This result is definitly capable of still further improvement by consideration of sparse crossbars which are "irregular." The proof of the analog of Lemma 5.2 then becomes very complicated, however, and the improvement is very slight.

A lower bound asymptotic to $3N \log_3 N$, applicable to both rearrangeable and non-blocking networks, is attributed to R. L. Dobrushin by Bassalygo and Pinsker [2]. A proof can be found by specializing the results in Pippenger [11] to $\epsilon = 0$; in fact, it follows from Lemma 3 and the obvious inequality $H(S) \geqslant \log_2 n! = n \log_2 n + O(n)$.

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